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On position and momentum operators in the q-oscillator

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Abstract
The position and momentum operators of the q-oscillator (with the main relation $aa^+ - qa^+a = 1$) are symmetric but not self-adjoint if $q > 1$. They have one-parameter family of self-adjoint extensions. These extensions are given explicitly. Their spectra and eigenfunctions are derived. Spectra of different extensions do not intersect. The results show that the creation and annihilation operators a^+ and a of the q-oscillator at $q > 1$ cannot determine a physical system without further more precise definition. In order to determine a physical system we have to choose appropriate self-adjoint extensions of the position and momentum operators.

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1. Introduction

A q-deformation of the quantum harmonic oscillator was introduced in [1–4]. There exist several variants of the q-oscillator. They are obtained from each other by some transformation (see [5]). One of the main problems for different forms of the q-oscillator is a form of spectra of the main operators, such as Hamiltonian, position and momentum operators. There is no problem with a spectrum of the Hamiltonian $H = \frac{1}{2}(aa^+ + a^+a)$. But in some cases there are difficulties with spectra of position and momentum operators (see, e.g., [6–8]). It was shown that if the position operator $Q = a^+ + a$ (or the momentum operator $P = \mathrm{i}(a^+ - a)$) is not bounded, then this symmetric operator is not essentially self-adjoint (see [6]). Moreover, in this case it has deficiency indices (1, 1), that is, it has a one-parameter family of self-adjoint extensions. Finding of self-adjoint extensions of a closed symmetric (but not self-adjoint) operator is a complicated problem. But we need to know self-adjoint extensions in order to be able to find their spectra (it does not make sense to talk about spectra of the operators Q and P if they are not self-adjoint).

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The main aim of this paper is to study self-adjoint extensions of the position and momentum operators Q and P for the q-oscillator with the main relation

$$aa^+ - qa^+a = 1$$

when $q > 1$. For these values of q the operators Q and P are unbounded and not essentially self-adjoint (for $q < 1$, these operators are bounded and, therefore, self-adjoint; they are studied in [7]). The operators Q and P can be represented in an appropriate basis by a Jacobi matrix. This means that they can be studied by means of properties of q-orthogonal polynomials associated with them. These q-orthogonal polynomials are expressed in terms of q^{-1}-continuous Hermite polynomials $h_n(x|q)$ introduced by Askey [9]. These polynomials correspond to an indeterminate moment problem and, therefore, are orthogonal with respect to infinitely many positive measures. Using orthogonality measures for these polynomials we shall find spectra of self-adjoint extensions of the operators Q and P. As we shall see, different self-adjoint extensions have different spectra.

For this reason we may make the following conclusion: the creation and annihilation operators a^+ and a of the q-oscillator at $q > 1$ do not determine uniquely a physical system. In order to fix a physical system we have to choose appropriate self-adjoint extensions of the position and momentum operators. This conclusion must be taken into account under applications of q-oscillators with $q > 1$ (e.g., under studying q-bose gas or under application in quantum field theory). We cannot operate with the creation and annihilation operators as freely as in the case of the usual quantum harmonic oscillator.

As far as we know the self-adjoint extensions found in this paper is the first example of all such extensions being found for a symmetric operator in an explicit form.

Below we use (without additional explanation) notations of the theory of q-special functions (see [10]). In order to study the position and momentum operators Q and P we shall need the results on Jacobi matrices, orthogonal polynomials and symmetric operators representable by a Jacobi matrix. In the next section, we give a combined exposition of some results on this connection from the books [11], chapter VII, [12] and from the paper [13] in a form appropriate for use below, and some consequences of them.

2. Jacobi matrices, orthogonal polynomials and moment problem

Operators, studied in this paper, are symmetric operators, representable by a Jacobi matrix. By a symmetric Jacobi matrix we mean a (finite or infinite) symmetric matrix of the form

$$M = \begin{pmatrix} b_0 & a_0 & 0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & 0 & a_2 & b_3 & a_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \tag{1}$$

We assume that $a_i \neq 0$, $i = 0, 1, 2, \ldots$. Let L be a closed symmetric operator on a Hilbert space $\mathcal{V}$, representable in some basis by a Jacobi matrix M. Then there exists an orthonormal basis e_n, $n = 0, 1, 2, \ldots$, of $\mathcal{V}$ such that

$$Le_n = a_n e_{n+1} + b_n e_n + a_{n-1} e_{n-1},$$

where $e_{-1} \equiv 0$. Let $|x\rangle = \sum_{n=0}^{\infty} p_n(x) e_n$ be an eigenvector of L with an eigenvalue x, $L|x\rangle = x|x\rangle$. Then

$$L|x\rangle = \sum_{n=0}^{\infty} [p_n(x) a_n e_{n+1} + p_n(x) b_n e_n + p_n(x) a_{n-1} e_{n-1}] = x \sum_{n=0}^{\infty} p_n(x) e_n.$$

Equating coefficients of the vector e_n, one comes to a recurrence relation for the coefficients $p_n(x)$:

$$a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1} p_{n-1}(x) = x p_n(x). \tag{2}$$

Since $p_{-1}(x) = 0$, by setting $p_0(x) \equiv 1$ we see that this relation recursively determines the coefficients $p_n(x)$. Moreover, a recurrence computation of $p_n(x)$ shows that these coefficients $p_n(x)$ are polynomials in x of degrees n, respectively. Since the coefficients a_n and b_n are real (because the operator L is symmetric), all coefficients of the polynomials $p_n(x)$ are real. Therefore conditions of Favard's characterization theorem (see, e.g., [14]) for the polynomials $p_n(x)$ are satisfied because in the case under discussion they simply reduce to the inequalities $a_n^2 > 0$ $n = 0, 1, 2, \ldots$. This means that these polynomials are orthogonal with respect to some positive measure $\mu(x)$ (which may be concentrated on a countable set of points).

It is known that orthogonal polynomials admit orthogonality with respect to either unique positive measure or with respect to infinitely many positive measures. One says that in the first case the polynomials correspond to the determinate moment problem and in the second case to the indeterminate (Hamburger) moment problem.

The polynomials $p_n(x)$ are very important for studying properties of the closed symmetric operator L. Namely, the following statements are true:

I. If the polynomials $p_n(x)$ are orthogonal with respect to a unique orthogonality measure μ,

$$\int p_m(x) p_n(x)\, \mathrm{d}\mu(x) = h_n \delta_{mn}, \qquad h_n > 0,$$

where the integration is performed over some subset (possibly discrete) of $\mathbb{R}$, then the *closed operator L is self-adjoint*. Moreover, a spectrum of the operator L is simple and coincides with the set, on which the polynomials $p_n(x)$ are orthogonal (recall that we assume that all numbers a_n are non-vanishing). The measure $\mu(x)$ determines the spectral measure for the operator L (for details see [11], chapter VII).

II. Let the polynomials $p_n(x)$ be orthogonal with respect to infinitely many orthogonality measures μ. Then a closed symmetric operator L is not self-adjoint and has deficiency indices $(1, 1)$, that is, it has infinitely many (in fact, a one-parameter family of) self-adjoint extensions. It is known that among orthogonality measures, with respect to which the polynomials are orthogonal, there are so-called extremal measures (that is, such measures that a set of polynomials $\{p_n(x)\}$ is complete in the Hilbert space L^2 with respect to the corresponding orthogonality measure). These measures uniquely determine self-adjoint extensions of the symmetric operator L. There exists a one-to-one correspondence between essentially different extremal orthogonality measures and self-adjoint extensions of the operator L. The extremal orthogonality measures determine spectra of the corresponding self-adjoint extensions. A spectrum of an self-adjoint extension coincides with the set on which the corresponding orthogonality measure is concentrated.

The inverse statements are also true:

I′. Let the operator L be self-adjoint. Then the corresponding polynomials $p_n(x)$ are orthogonal with respect to a unique orthogonality measure μ,

$$\int p_m(x) p_n(x)\, \mathrm{d}\mu(x) = h_n \delta_{mn}, \qquad h_n > 0,$$

where the integral is taken over some subset (possibly discrete) of $\mathbb{R}$. Moreover, a measure μ is uniquely determined by a spectrum of L (for details see [11], chapter VII).

II′. Let the closed symmetric operator L be not self-adjoint. Since it is representable by a Jacobi matrix (1) with $a_n \neq 0$, $n = 0, 1, 2, \ldots$, it admits a one-parameter family of self-adjoint extensions. Then the polynomials $p_n(x)$ are orthogonal with respect to infinitely many orthogonality measures μ. Moreover, spectral measures of self-adjoint extensions of L determine extremal orthogonality measures for the polynomials $\{p_n(x)\}$ (a set of polynomials $\{p_n(x)\}$ is complete in the Hilbert spaces $L^2(\mu)$ with respect to the corresponding measures μ).

To a set of orthogonal polynomials $p_n(x), n = 0, 1, 2, \ldots,$ associated with an indeterminate (Hamburger) moment problem, there uniquely correspond four entire functions $A(z), B(z), C(z), D(z)$, which are related to appropriate orthogonality measures μ by the formula

$$F(z) \equiv \frac{A(z) - \sigma(z)C(z)}{B(z) - \sigma(z)D(z)} = \int_{-\infty}^{\infty} \frac{\mathrm{d}\mu(t)}{z - t} \tag{3}$$

(see, e.g., [12]), where $\sigma(z)$ is any analytic function. Moreover, to each analytic function $\sigma(z)$ (including constants and $\sigma = \pm\infty$) there corresponds a single orthogonality measure $\mu_\sigma(z)$ and, conversely, to each orthogonality measure μ there corresponds an analytic function σ such that formula (3) holds. There exists the Stieltjes inversion formula, which converts formula (3) (see [12]).

Thus, orthogonality measures for a given set of polynomials $p_n(x), n = 0, 1, 2, \ldots,$ in principle, can be found. However, it is very difficult to evaluate the functions $A(z), B(z), C(z), D(z)$. In [15] they are evaluated for a particular example of polynomials, namely, for the q^{-1}-continuous Hermite polynomials $h_n(x|q)$.

The set of extremal measures coincides with the set of the measures $\mu(t) \equiv \mu_\sigma(t)$, corresponding to constants σ (including $\sigma = \pm\infty$). All other orthogonality measures are not extremal. Extremal orthogonality measures have the following properties.

(a) If $\mu_\sigma(x)$ is an extremal measure, corresponding (according to formula (3)) to a number σ, then $\mu_\sigma(x)$ is a step function. A spectrum of $\mu_\sigma(x)$ (that is, the set on which this measure is concentrated) coincides with the set of zeros of the denominator $B(z) - \sigma D(z)$ from (3). The mass, concentrated in a spectral point x_j (that is, a jump of $\mu_\sigma(x)$ in the point x_j), is equal to $\left(\sum_{n=0}^{\infty} |p_n(x_j)|^2\right)^{-1}$.
(b) Spectra of extremal measures are real and simple.
(c) Spectral points of two different extremal measures $\mu_\sigma(x)$ and $\mu_{\sigma'}(x)$ are mutually separated.
(d) For a given real number x_0, there exists a (unique) real number σ, such that the measure $\mu_\sigma(x)$ has x_0 as its spectral point. The points of spectrum of $\mu_\sigma(x)$ are analytic monotonic functions of σ.

Since spectra of self-adjoint extensions of the operator L coincide with the spectra of the corresponding extremal orthogonal measures for the polynomials $p_n(x)$, then properties (a)–(d) can be formulated for spectra of these self-adjoint extensions:

(a′) Spectra of self-adjoint extensions of L are discrete.
(b′) Self-adjoint extensions of L have simple spectra, that is, spectral points are not multiple.
(c′) Spectra of two different self-adjoint extensions of L are mutually separated.
(d′) For a given real number x_0, there exists a (unique) self-adjoint extension L^{ext} such that x_0 is a spectral point of L^{ext}.

3. The *q*-oscillator

A q-deformation of the quantum harmonic oscillator was given in [1, 2]. In 1989, Biedenharn [3] and Macfarlane [4] defined the q-oscillator in more exact form. For our definition of the

q-oscillator we use the following relations:

$$aa^+ - qa^+a = 1, \qquad [N, a^+] = a^+, \qquad [N, a] = -a \tag{4}$$

for the creation and annihilation operators a^+, a and for the number operator N.

The Fock representation of this q-oscillator acts on the Hilbert space $\mathcal{H}$ with the orthonormal basis $|n\rangle$, $n = 0, 1, 2, \ldots$, and is given by the formulae

$$a|n\rangle = \{n\}_q^{1/2}|n-1\rangle, \qquad a^+|n\rangle = \{n+1\}_q^{1/2}|n+1\rangle, \qquad N|n\rangle = n|n\rangle, \tag{5}$$

where

$$\{n\}_q := \frac{q^n - 1}{q - 1}.$$

We shall use the following functional realization of the Hilbert space $\mathcal{H}$. Let $\mathcal{P}$ be the space of all polynomials in a real variable y. We introduce in $\mathcal{P}$ a scalar product such that the monomials

$$e_n \equiv e_n(y) := \frac{(-1)^{n/2}}{(q;q)_n^{1/2}} y^n, \tag{6}$$

where $(b; q)_n := (1-b)(1-bq)\cdots(1-bq^{n-1})$, constitute an orthonormal basis of $\mathcal{P}$. The orthonormality of this basis gives a scalar product in $\mathcal{P}$. We close the space $\mathcal{P}$ with respect to this scalar product and obtain a Hilbert space which can be considered as a realization of the Hilbert space $\mathcal{H}$. The operators a^+ and a are realized on this space as

$$a^+ = (q-1)^{-1/2} y, \qquad a = (q-1)^{1/2} D_q,$$

where D_q is the q-derivative determined by

$$D_q f(y) = \frac{f(qy) - f(y)}{(q-1)y}.$$

Then operators a^+ and a act upon the basis elements (6) by formulae (5). Everywhere below we assume that $\mathcal{H}$ is the Hilbert space of functions in y, introduced above.

We are interested in the position and momentum operators

$$Q = a^+ + a, \qquad P = \mathrm{i}(a^+ - a)$$

of the q-oscillator (4). We have the formulae

$$Qe_n = \{n\}_q^{1/2} e_{n-1} + \{n+1\}_q^{1/2} e_{n+1}, \tag{7}$$

$$Pe_n = \mathrm{i}\{n\}_q^{1/2} e_{n-1} - \mathrm{i}\{n+1\}_q^{1/2} e_{n+1}, \tag{8}$$

which follow from (5). It is clear from these formulae that Q and P are unbounded symmetric operators. It is known (see, e.g., [6]) that under $q > 1$ closures of these operators are not self-adjoint, but each of them has a one-parameter family of self-adjoint extensions. One of the aims of this paper is to give these self-adjoint extensions.

4. Eigenfunctions of the position operator

We suppose below that q is a fixed real number such that $q > 1$. We also introduce the notation $\breve{q} = q^{-1}$.

The aim of this section is to derive formulae for eigenfunctions $\varphi_x(y)$ of the position operator Q:

$$Q\varphi_x(y) = x\varphi_x(y).$$

Let us show that

$$\varphi_x(y) = \prod_{n=0}^{\infty}(1 + 2yx'\breve{q}^{n+1} - y^2\breve{q}^{2n+2}), \tag{9}$$

where $x' := \frac{1}{2}(q-1)^{1/2}x$. Using the definition of the q-derivative D_q we have

$$D_q\varphi_x(y) = \frac{\varphi_x(qy) - \varphi_x(y)}{y(q-1)} = \frac{2x'-y}{q-1}\varphi_x(y).$$

Therefore,

$$Q\varphi_x(y) = \left\{\frac{y}{(q-1)^{1/2}} + \frac{2x'-y}{(q-1)^{1/2}}\right\}\varphi_x(y) = \frac{2x'}{(q-1)^{1/2}} = x\varphi_x(y)$$

that is, functions (9) are eigenfunctions of the operator Q.

Let us reduce functions (9) to another form. To do this we note that

$$1 + 2yx'\breve{q}^{n+1} - y^2\breve{q}^{2n+2} = \big(1 + \breve{q}^n y\breve{q}\big(\sqrt{1+x'^2} + x'\big)\big)\big(1 - \breve{q}^n y\breve{q}\big(\sqrt{1+x'^2} - x'\big)\big).$$

Thus,

$$\begin{aligned}\varphi_x(y) &= \prod_{n=0}^{\infty}\big(1 + \breve{q}^n y\breve{q}\big(\sqrt{1+x'^2} + x'\big)\big)\big(1 - \breve{q}^n y\breve{q}\big(\sqrt{1+x'^2} - x'\big)\big) \\ &= \big(-y\breve{q}\big(\sqrt{1+x'^2} + x'\big); \breve{q}\big)_\infty \big(y\breve{q}\big(\sqrt{1+x'^2} - x'\big); \breve{q}\big)_\infty.\end{aligned}$$

Comparing the right-hand side with the right-hand side in the formula

$$\sum_{n=0}^{\infty}\frac{t^n\breve{q}^{n(n-1)/2}}{(\breve{q};\breve{q})_n}h_n(y|\breve{q}) = \big(-t\big(\sqrt{y^2+1} + y\big); \breve{q}\big)_\infty \big(t\big(\sqrt{y^2+1} - y\big); \breve{q}\big)_\infty$$

(see formula (2.4) in [15]), giving a generating function for the q^{-1}-Hermite polynomials $h_n(x|\breve{q})$ defined by

$$h_n(x|\breve{q}) = \sum_{k=0}^{n}\frac{(-1)^k\breve{q}^{k(k-n)}(\breve{q};\breve{q})_n}{(\breve{q};\breve{q})_k(\breve{q};\breve{q})_{n-k}}\big(\sqrt{x^2+1} + x\big)^{n-2k},$$

we conclude that the functions $\varphi_x(y)$ can be decomposed in the orthogonal polynomials $h_n(x'|\breve{q})$. We have

$$\varphi_x(y) = \sum_{n=0}^{\infty}\frac{y^n\breve{q}^{n(n+1)/2}}{(\breve{q};\breve{q})_n}h_n(x'|\breve{q}). \tag{10}$$

Taking into account expression (6) for the basis elements e_n and the formula

$$(\breve{q};\breve{q})_n = (-1)^n(q;q)_n q^{-n(n+1)/2}, \qquad \breve{q} = q^{-1},$$

we derive that

$$\begin{aligned}\varphi_x(y) &= \sum_{n=0}^{\infty}\frac{(-1)^n y^n}{(q;q)_n}h_n(x'|\breve{q}) = \sum_{n=0}^{\infty}\frac{(-1)^n(-1)^{-n/2}e_n(y)}{(q;q)_n^{1/2}}h_n(x'|\breve{q}) \\ &= \sum_{n=0}^{\infty}e_n(y)\frac{(-1)^n\breve{q}^{n(n+1)/4}}{(\breve{q};\breve{q})_n^{1/2}}h_n(x'|\breve{q}).\end{aligned}$$

Thus, we proved the following decomposition of the eigenfunctions $\varphi_x(y)$ in the basis elements (6) of the Hilbert space $\mathcal{H}$:

$$\varphi_x(y) = \sum_{n=0}^{\infty}P_n(x)e_n(y), \tag{11}$$

where the coefficients $P_n(x)$ are given by the formula

$$P_n(x) = (-1)^n \check{q}^{n(n+1)/4} (\check{q};\check{q})_n^{-1/2} h_n(x'|\check{q}), \tag{12}$$

where, as before, $x' = \frac{1}{2}(q-1)^{1/2}x$.

We have found that eigenfunctions of the position operator Q are given by formula (10). However, we do not know the spectra of self-adjoint extensions Q^{ext} of Q. In order to find these extensions and their spectra we use assertions of section 2. Namely, since eigenfunctions of Q are expressed in terms of the basis elements $e_n(y)$ by formula (11), then self-adjoint extensions Q^{ext} and their spectra are determined by orthogonality relations of the polynomials (12).

5. Spectra of self-adjoint extensions of Q

The polynomials $h_n(z|\check{q}), n = 0, 1, 2, \ldots, 0 < \check{q} < 1$, have infinitely many orthogonality relations. Extremal orthogonality measures are parametrized by a real number $b, \check{q} \leqslant b < 1$, which is related to the parameter σ of section 2 (see [15]). It is shown in [15] that for a fixed b, the corresponding orthogonality measure is concentrated on the discrete set of points

$$z_r(b) = \tfrac{1}{2}(\check{q}^{-r}b^{-1} - b\check{q}^r), \qquad r = 0, \pm1, \pm2, \ldots, \tag{13}$$

and the orthogonality relation is given by

$$\sum_{r=-\infty}^{\infty} m_r h_n(z_r(b)|\check{q}) h_{n'}(z_r(b)|\check{q}) = \check{q}^{-n(n+1)/2}(\check{q};\check{q})_n \delta_{nn'}, \tag{14}$$

where the weight function m_r coincides with

$$m_r = \frac{b^{4r}\check{q}^{r(2r-1)}(1+b^2\check{q}^{2r})}{(-b^2;\check{q})_\infty(-\check{q}/b^2;\check{q})_\infty(\check{q};\check{q})_\infty}. \tag{15}$$

Therefore, the orthogonality relations for the polynomials (12) with extremal orthogonality measures are given by the same parameter $b, \check{q} \leqslant b < 1$, and for fixed b the measure is concentrated on the discrete set

$$x_r(b) = (\check{q}^{-r}b^{-1} - b\check{q}^r)/(q-1)^{1/2}, \qquad r = 0, \pm1, \pm2, \ldots. \tag{16}$$

The corresponding orthogonality relation is

$$\sum_{r=-\infty}^{\infty} m_r P_n(x_r(b)) P_{n'}(x_r(b)) = \delta_{nn'}, \tag{17}$$

where m_r is given by (15).

These orthogonality relations and assertions of section 2 allow us to make the following conclusions. *Self-adjoint extensions Q_b^{ext} of the position operator Q are given by the parameter $b, \check{q} \leqslant b < 1$. Moreover, the spectrum of the extension Q_b^{ext} coincides with the set of points*

$$x_r(b) = (q^r b^{-1} - bq^{-r})/(q-1)^{1/2}, \qquad r = 0, \pm1, \pm2, \ldots. \tag{18}$$

These points coincide with the values of the coordinate of our physical system fixed by the parameter b. To the eigenvalues (18) there correspond eigenfunctions

$$\varphi_{x_r(b)}(y) = \sum_{n=0}^{\infty} P_n(x_r(b)) e_n(y), \qquad r = 0, \pm1, \pm2, \ldots. \tag{19}$$

It follows from (18) and from assertions of section 2 that

(a) spectra of the self-adjoint extensions Q_b^{ext} are discrete and simple;
(b) spectra of two different self-adjoint extensions Q_b^{ext} and $Q_{b'}^{\text{ext}}, b \neq b'$, are mutually separated;

(c) for a given real number x_0 there exists a (unique) self-adjoint extension $Q_b^{\rm ext}$ such that x_0 is a spectral point of $Q_b^{\rm ext}$.

For a fixed b, eigenfunctions (19) are linearly independent (and, therefore, orthogonal), since they correspond to different eigenvalues of $Q_b^{\rm ext}$. Since the corresponding orthogonality measure in (17) is extremal, they constitute a basis of the Hilbert space $\mathcal{H}$. Let us normalize these basis elements. To do this, we have to multiply each $\varphi_{x_r(b)}(y)$ by the corresponding normalization constant:

$$\varphi_{x_r(b)}^{\rm norm}(y) = c_r(b)\varphi_{x_r(b)}(y), \qquad r = 0, \pm 1, \pm 2, \ldots.$$

These functions form an orthonormal basis of $\mathcal{H}$. Since

$$\varphi_{x_r(b)}^{\rm norm}(y) = \sum_{n=0}^{\infty} c_r(b) P_n(x_r(b)) e_n(y)$$

the matrix $(a_{rn}), a_{rn} = c_r(b)P_n(x_r(b))$, where $r = 0, \pm 1, \pm 2, \ldots$ and $n = 0, 1, 2, \ldots,$ connects two orthonormal bases of the Hilbert space $\mathcal{H}$. Therefore, this matrix is unitary, that is,

$$\sum_{r=-\infty}^{\infty} |c_r(b)|^2 P_n(x_r(b)) P_{n'}(x_r(b)) = \delta_{nn'}.$$

Comparing this formula with relation (17) we have $c_r(b) = m_r^{1/2}$ and

$$\varphi_{x_r(b)}^{\rm norm}(y) = m_r^{1/2}\varphi_{x_r(b)}(y), \qquad r = 0, \pm 1, \pm 2, \ldots, \tag{20}$$

where $m_r \equiv m_r(b)$ is given by (15).

In order to realize $Q_b^{\rm ext}$ as a self-adjoint operator, we construct a one-to-one isometry Ω of the Hilbert space $\mathcal{H}$ onto the Hilbert space $L_b^2(m_r)$ of functions F on the set of points (18) with the scalar product

$$\langle F(x_r(b)), F'(x_r(b))\rangle = \sum_{r=-\infty}^{\infty} m_r F(x_r(b))\overline{F'(x_r(b))}.$$

It follows from (17) that the polynomials $P_n(x_r(b))$ are orthogonal on the set (18) and constitute an orthonormal basis of $L_b^2(m_r)$. For a fixed b, the isometry Ω is given by the formula

$$\Omega : \mathcal{H} \ni f \to F(x_r(b)) := m_r^{-1/2}\big\langle f, \varphi_{x_r(b)}^{\rm norm}(y)\big\rangle \in L_b^2(m_r).$$

It follows from (20) that

$$\mathcal{H} \ni e_n(y) \to m_r^{-1/2}\big\langle e_n(y), \varphi_{x_r(b)}^{\rm norm}(y)\big\rangle = P_n(x_r(b)).$$

This formula shows that Ω is indeed a one-to-one isometry.

The operator $Q_b^{\rm ext}$ acts on $L_b^2(m_r)$ as the multiplication operator:

$$Q_b^{\rm ext} F(x_r) = x_r(b)F(x_r).$$

It is known (see [16]) that the multiplication operator is a self-adjoint operator.

The Hilbert space $L_b^2(m_r)$ is the space of states of our physical system in the coordinate representation. Since the elements $e_n(y) \in \mathcal{H}$ are eigenfunctions of the Hamiltonian $H = \frac{1}{2}(aa^+ + a^+a)$, then $P_n(x_r) \in \hat{L}_b^2(m_r)$ are eigenfunctions of the same Hamiltonian if its action is considered on $L_b^2(m_r)$.

Recall that for different values of b the sets (18) of values of the coordinate are different. Therefore, the spaces $L_b^2(m_r)$ for them are different, since they consist of functions defined on different sets.

6. Eigenfunctions and spectra of the momentum operator

By changing the basis $\{e_n(y)\}$ by the basis $\{e'_n(y)\}$, where $e'_n(y) = \mathrm{i}^{-n} e_n(y)$, we see that the momentum operator $P = \mathrm{i}(a^+ - a)$ is given in the last basis by the same formula as the position operator is given in the basis $\{e_n(y)\}$. This means that the operator P is symmetric, but is not self-adjoint. Moreover, it has infinitely many (in fact, a one-parameter family of) self-adjoint extensions.

Eigenfunctions of the momentum operator can be found (by using the basis $\{e'_n(y)\}$) in the same way as in the case of the position operator. For this reason, we give only the results.

Eigenfunctions $\xi_p(y)$ of the momentum operator P, $P\xi_p(y) = p\xi_p(y)$, are of the form

$$\xi_p(y) = \prod_{n=0}^{\infty}(1 - 2\mathrm{i}yp'\breve{q}^{n+1} + y^2\breve{q}^{2n+2})$$
$$= \big(\mathrm{i}y\breve{q}\big(\sqrt{1+p'^2} + p'\big); \breve{q}\big)_\infty \big(-\mathrm{i}y\breve{q}\big(\sqrt{1+p'^2} - p'\big); \breve{q}\big)_\infty,$$

where $p' := \frac{1}{2}(q-1)^{1/2}p$. The function $\xi_p(y)$ can be decomposed in the q^{-1}-Hermite polynomials $h_n(p|\breve{q})$:

$$\xi_p(y) = \sum_{n=0}^{\infty} \frac{\mathrm{i}^{-n} y^n \breve{q}^{n(n+1)/2}}{(\breve{q}; \breve{q})_n} h_n(p'|\breve{q}) = \sum_{n=0}^{\infty} e_n(y) \frac{\mathrm{i}^n \breve{q}^{n(n+1)/4}}{(\breve{q}; \breve{q})_n^{1/2}} h_n(x'|\breve{q}).$$

Thus, we have the following decomposition of the eigenfunctions $\xi_p(y)$ in the basis elements (6) of the Hilbert space $\mathcal{H}$:

$$\xi_p(y) = \sum_{n=0}^{\infty} \tilde{P}_n(x) e_n(y), \qquad (21)$$

where the coefficients $\tilde{P}_n(x)$ are given by the formula

$$\tilde{P}_n(x) = \frac{\mathrm{i}^n \breve{q}^{n(n+1)/4}}{(\breve{q}; \breve{q})_n^{1/2}} h_n(x'|\breve{q}).$$

Using orthogonality relations for the polynomials $h_n(x'|\breve{q})$, described above, we conclude that *self-adjoint extensions* P_b^{ext} *of the position operator* P *are given by the parameter* b, $\breve{q} \leqslant b < 1$. *The spectrum of the extension* P_b^{ext} *coincides with the set of points*

$$p_r(b) = (q^r b^{-1} - bq^{-r})/(q-1)^{1/2}, \qquad r = 0, \pm 1, \pm 2, \ldots. \qquad (22)$$

This set coincides with the set of values of the momentum of our physical system fixed by the parameter b. To the eigenvalues (22) there correspond eigenfunctions

$$\xi_{p_r(b)}(y) = \sum_{n=0}^{\infty} \tilde{P}_n(p_r(b)) e_n(y), \qquad r = 0, \pm 1, \pm 2, \ldots. \qquad (23)$$

It follows from (22) and from assertions of section 2 that

(a) spectra of the self-adjoint extensions P_b^{ext} are discrete and simple;
(b) spectra of two different self-adjoint extensions P_b^{ext} and $P_{b'}^{\mathrm{ext}}$, $b \neq b'$, are mutually separated;
(c) for a given real number p_0 there exists a (unique) self-adjoint extension P_b^{ext} such that p_0 is a spectral point of P_b^{ext}.

In the same way as in the case of the position operator we derive that eigenfunctions (23) constitute a basis of the Hilbert space $\mathcal{H}$. Let us normalize it. To do this, we make the same reasoning as in section 5, and obtain that the functions

$$\xi^{\text{norm}}_{p_r(b)}(y) = m_r^{1/2}\xi_{p_r(b)}(y), \qquad r = 0, \pm 1, \pm 2, \ldots,$$

form a normalized basis of $\mathcal{H}$, where $m_r \equiv m_r(b)$ is given by (15).

In order to realize P_b^{ext} as a self-adjoint operator we again use the reasoning of section 5, namely, we construct a one-to-one isometry Ω' of the Hilbert space $\mathcal{H}$ onto the Hilbert space $\hat{L}^2_b(m_r)$ of functions F on the set of points (22) with the scalar product

$$\langle F(p_r(b)), F'(p_r(b))\rangle = \sum_{r=-\infty}^{\infty} m_r F(p_r(b))\overline{F'(p_r(b))}.$$

It follows from (17) that the polynomials $\tilde{P}_n(p_r(b))$ are orthogonal and constitute an orthonormal basis of $\hat{L}^2_b(m_r)$. For a fixed b, the isometry Ω' is given by the formula

$$\Omega' : \mathcal{H} \ni f \to F(p_r(b)) = m_r^{-1/2}\big\langle f, \xi^{\text{norm}}_{p_r(b)}(y)\big\rangle \in \hat{L}^2_b(m_r).$$

It follows from (21) that

$$\mathcal{H} \ni e_n(y) \to m_r^{-1/2}\big\langle e_n(y), \xi^{\text{norm}}_{p_r(b)}(y)\big\rangle = \tilde{P}_n(p_r(b)).$$

This formula shows that Ω' is indeed a one-to-one isometry.

The operator P_b^{ext} acts on $\hat{L}^2_b(m_r)$ as the multiplication operator: $P_b^{\text{ext}}F(p_r(b)) = p_r(b)F(p_r(b))$ and this operator is self-adjoint.

The Hilbert space $\hat{L}^2_b(m_r)$ is the space of states of our physical system in the momentum representation. Since the elements $e_n(y) \in \mathcal{H}$ are eigenfunctions of the Hamiltonian $H = \frac{1}{2}(aa^+ + a^+a)$, then $\tilde{P}_n(p_r(b)) \in \hat{L}^2_b(m_r)$ are eigenfunctions of the same Hamiltonian if its action is considered in $\hat{L}^2_b(m_r)$.

Recall that for different values of b the sets (22) of values of the momentum are different. Therefore, the spaces $\hat{L}^2_b(m_r)$ for them are different, since they consist of functions defined on different sets. Clearly, we may identify $L^2_b(m_r)$ with $\hat{L}^2_b(m_r)$.

Our consideration shows that the creation and annihilation operators a^+ and a of section 3 at $q > 1$ cannot determine a physical system without further indications. Namely, in order to determine a physical system we have to take appropriate self-adjoint extensions of the operators Q and P. Thus, the q-oscillator algebra of section 3 in fact determine two-parameter family of q-oscillators. We denote them by $O(b, b')$, $\breve{q} \leqslant b, b' < 1$, where b and b' are determined by Q_b^{ext} and $P_{b'}^{\text{ext}}$.

7. The related Fourier transforms

It is well known that the Fourier transform gives for the usual quantum harmonic oscillator a transition from the position space to the momentum space and vice versa. An analogue of the Fourier transform for the q-oscillator in the case when $0 < q < 1$ is derived in [17]. The aim of this section is to give an analogue of the Fourier transform for the q-oscillator $O(b, b')$ for fixed b and b'.

We fix b and b' from the interval $[\breve{q}, 1)$. Let $f \in \mathcal{H}$ and

$$\Omega f = F(x_r(b)) \in L^2_b(m_r), \qquad \Omega' f = \hat{F}(p_{r'}(b')) \in \hat{L}^2_{b'}(m_{r'}).$$

We have to find a linear transform $\mathcal{F} : \hat{L}^2_{b'}(m_{r'}) \to L^2_b(m_r)$ such that $\mathcal{F}\hat{F} = F$. By the definition of Ω and Ω', one has

$$F(x_r(b)) = m_r^{-1/2}(b)\big\langle f, \varphi^{\text{norm}}_{x_r(b)}\big\rangle, \qquad \hat{F}(p_{r'}(b')) = m_{r'}^{-1/2}(b')\big\langle f, \xi^{\text{norm}}_{p_{r'}(b')}\big\rangle.$$

It is clear that

$$\varphi^{\mathrm{norm}}_{x_r(b)}(y) = \sum_{r'=-\infty}^{\infty} \langle \xi^{\mathrm{norm}}_{p_{r'}(b')}, \varphi^{\mathrm{norm}}_{x_r(b)} \rangle \xi^{\mathrm{norm}}_{p_{r'}(b')}(y) = \sum_{r'=-\infty}^{\infty} \mathcal{F}^{b'b}_{r'r} \xi^{\mathrm{norm}}_{p_{r'}(b')}(y),$$

where $\mathcal{F}^{b'b}_{r'r} = \langle \xi^{\mathrm{norm}}_{p_{r'}(b')}, \varphi^{\mathrm{norm}}_{x_r(b)} \rangle$. Therefore,

$$F(x_r(b)) = m_r^{-1/2}(b) \sum_{r'=-\infty}^{\infty} \mathcal{F}^{b'b}_{r'r} \langle f, \xi^{\mathrm{norm}}_{p_{r'}(b')} \rangle = \left(\frac{m_{r'}(b')}{m_r(b)} \right)^{1/2} \sum_{r'=-\infty}^{\infty} \mathcal{F}^{b'b}_{r'r} \hat{F}(p_{r'}(b')). \tag{24}$$

Thus, an analogue of the Fourier transform for the q-oscillator $O(b, b')$ is given by the unitary matrix $\left(\mathcal{F}^{b'b}_{r'r}\right)^{\infty}_{r',r=-\infty}$. For entries of this matrix we have

$$\begin{aligned}\mathcal{F}^{b'b}_{r'r} &= \langle \xi^{\mathrm{norm}}_{p_{r'}(b')}, \varphi^{\mathrm{norm}}_{x_r(b)} \rangle = M^{1/2} \langle \xi_{p_{r'}}(b'), \varphi_{x_r(b)} \rangle = M^{1/2} \sum_{n=0}^{\infty} \tilde{P}_n(p_{r'}(b')) P_n(x_r(b)) \\ &= M^{1/2} \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n \breve{q}^{n(n+1)/2}}{(\breve{q}; \breve{q})_n} h_n \left(\frac{1}{2}(q^{r'} b'^{-1} - q^{-r'} b') \,|\breve{q} \right) h_n \left(\frac{1}{2}(q^r b^{-1} - q^{-r} b) |\breve{q} \right),\end{aligned}$$

where $M = m_{r'}(b') m_r(b)$. In order to sum up the last sum we set

$$q = e^{\tau}, \qquad b = e^{\sigma}, \qquad b' = e^{\sigma'}.$$

Then

$$\tfrac{1}{2}(q^{r'} b'^{-1} - q^{-r'} b') = \sinh(\tau r' - \sigma'), \qquad \tfrac{1}{2}(q^r b^{-1} - q^{-r} b) = \sinh(\tau r - \sigma).$$

Taking into account the relation

$$\sum_{n=0}^{\infty} h_n(\sinh \xi |\breve{q}) h_n(\sinh \eta |\breve{q}) \frac{\breve{q}^{n(n-1)/2}}{(\breve{q}; \breve{q})_n} R^n = \frac{(-Re^{\xi+\eta}, -Re^{-\xi-\eta}, Re^{\xi-\eta}, Re^{\eta-\xi}; \breve{q})_\infty}{(R^2/\breve{q}; \breve{q})_\infty},$$

where

$$(a, b, c, d; \breve{q})_m \equiv (a; \breve{q})_m (b; \breve{q})_m (c; \breve{q})_m (d; \breve{q})_M,$$

we derive that

$$\mathcal{F}^{b'b}_{r'r} = \frac{(\mathrm{i}\breve{q}^2 \exp^{(\tau(r'+r)-\sigma'-\sigma)}, \mathrm{i}\breve{q}^2 \exp^{(-\tau(r'+r)+\sigma'+\sigma)}, -\mathrm{i}\breve{q}^2 \exp^{(\tau(r'-r)+\sigma'-\sigma)}, -\mathrm{i}\breve{q}^2 \exp^{(\tau(r-r')-\sigma'+\sigma)}; \breve{q})_\infty}{M^{-1/2}(-\breve{q}^3; \breve{q})_\infty}. \tag{25}$$

Thus, the Fourier transform $\mathcal{F} \equiv \mathcal{F}^{b'b}$, corresponding to the q-oscillator $O(b, b')$, is given by formula (24), where the entries of the unitary matrix $\left(\mathcal{F}^{b'b}_{r'r}\right)^{\infty}_{r',r=-\infty}$ are determined by (25). The inverse transform $\mathcal{F}^{-1} F = \hat{F}$ is given by the inverse matrix $\left(\mathcal{F}^{b'b}_{r'r}\right)^{-1} \equiv \left(\mathcal{F}^{b'b}_{r'r}\right)^{*}$.

8. Concluding remarks

We studied the q-oscillator with the main relation $aa^+ - qa^+a = 1$ under $q > 1$. For these values of q, the position and momentum operators are symmetric but not self-adjoint. Each of them has a one-parameter family of self-adjoint extensions. We have found these self-adjoint extensions in an explicit form. These self-adjoint extensions have discrete spectra. Spectra of different self-adjoint extensions of the position operator do not intersect. The same assertion is true for the extensions of the momentum operator.

Our study show that the creation and annihilation operators a^+ and a of the q-oscillator algebra at $q > 1$ cannot determine a physical system without further more precise definition. Namely, in order to determine a physical system we have to choose appropriate self-adjoint extensions of the operators Q and P. This means that the q-oscillator algebra at $q > 1$ in fact determine two-parameter family of q-oscillators. These q-oscillators have different spectra of the position operator and different spectra of the momentum operator.

This conclusion must be taken into account under applications of q-oscillators with $q > 1$ (e.g., under studying q-bose gas or under application in quantum field theory). We cannot operate with the creation and annihilation operators as freely as we do in the case of the usual quantum harmonic oscillator. In a subsequent paper we shall apply the results of this paper for studying the corresponding q-bose gas.

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